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Ornstein Theory for Extended Symbolic Dynamics

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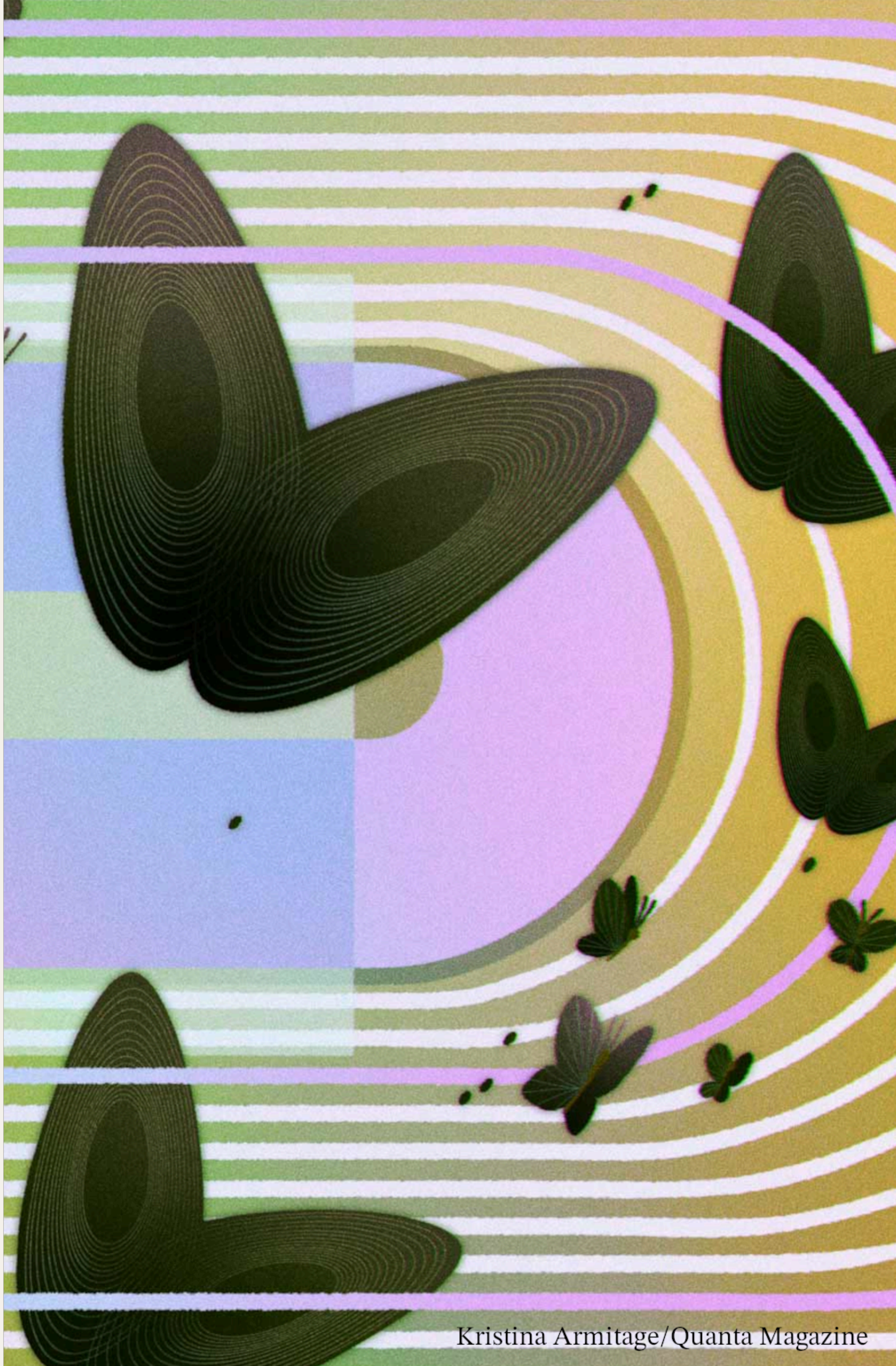
Introduction


Kristina Armitage/Quanta Magazine

Setup

$(X, \mathcal{B}, \mu)$ a probability space or a Lebesgue space.

$T : X \to X$ a measure-preserving transformation.

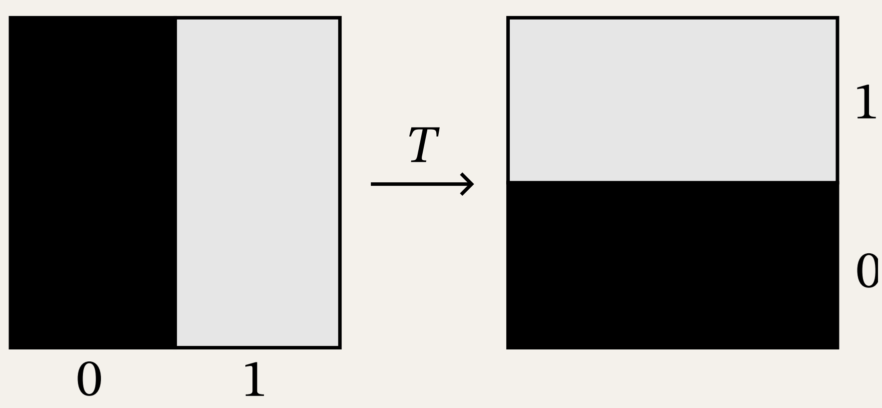
We say that $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ defined on $(X_1, \mathcal{B}_1, \mu)$ and $(X_2, \mathcal{B}_2, \nu)$, are *isomorphic* if there are $A_1 \in \mathcal{B}_1, A_2 \in \mathcal{B}_2$ such that

- $\mu(A_1) = \nu(A_2) = 1$
- $T_1(A_1) \subset A_1,\ T_2(A_2) \subset A_2$
- $\exists\, \varphi : A_1 \to A_2$ invertible measure preserving map such that

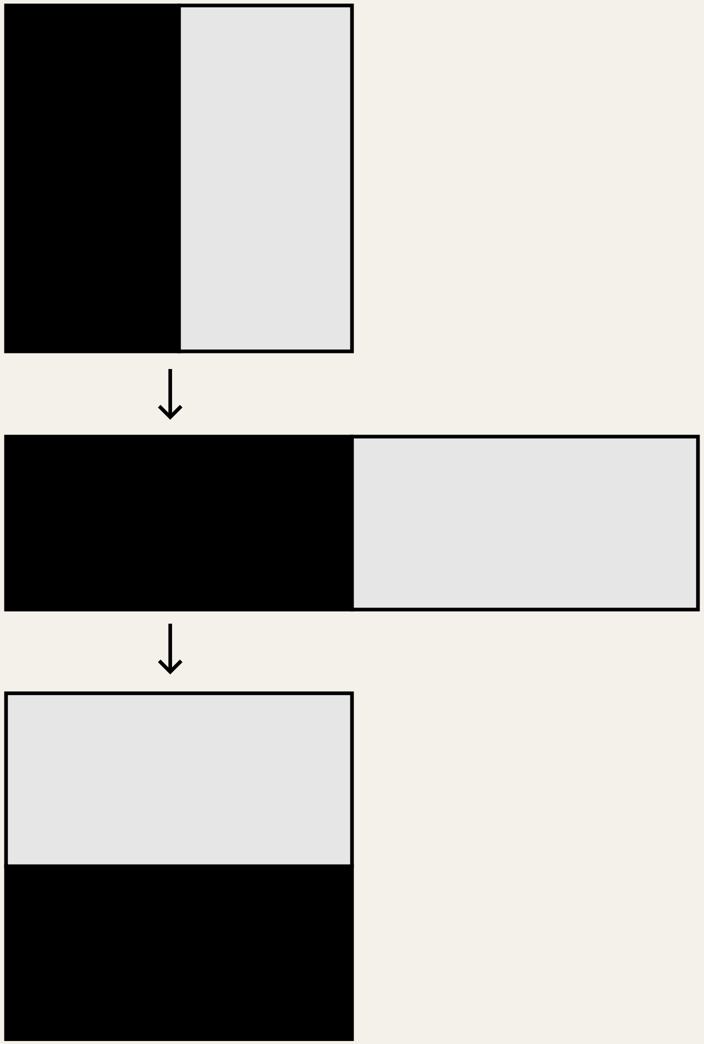
$$\varphi \circ T_1 = T_2 \circ \varphi.$$

Baker's map

$$X = [0,1]^2,\ T(x,y) = \begin{cases} (2x, y/2) & \text{if } x \in \left[0, \frac{1}{2}\right) \\ (2x-1, (y+1)/2) & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$



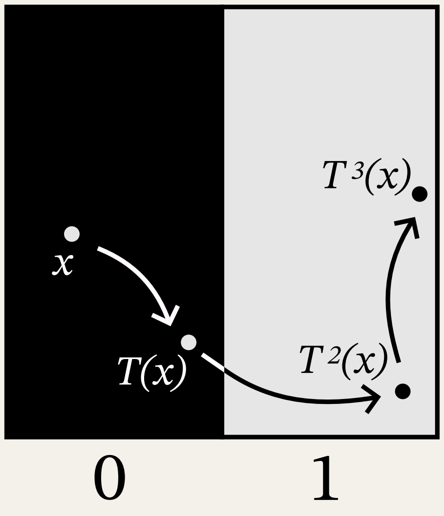
Baker's map



Coding the baker's map

To each orbit $\{\dots, T^{-1}(x), x, T(x), \dots\}$ we relate a sequence $(x_n)_n$ of zeros and ones:

- if $T^n x \in \left[0, \frac{1}{2}\right) \times [0,1]$, code $x_n = 0$
- if $T^n x \in \left[\frac{1}{2}, 1\right] \times [0,1]$, code $x_n = 1$



$$\mathcal{O}(x) = (\ \dots\ ;\ 0011\dots\ )$$
$$\mathcal{O}(T(x)) = (\ \dots 0\ ;\ 011\dots\ )$$

A is a finite alphabet

$\Sigma_{\mathrm{A}} = \mathrm{A}^{\mathbb{Z}} = \left\{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathrm{A}\right\}$

$$(x_n)_{n\in\mathbb{Z}} = (\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1 \cdots\ ) \in \Sigma_{\mathrm{A}}$$

Σ_{A} is a compact metric space with

$$d((x_n),(y_n)) = 2^{-\inf\{|i| : x_i \neq y_i\}}$$

The *Bernoullis shift* $\sigma : \Sigma_{\mathrm{A}} \to \Sigma_{\mathrm{A}}$ is the map

$$\sigma(\ \cdots x_{-2}x_{-1}\ ;\ x_0x_1 \cdots\ ) = (\ \cdots x_{-1}x_0\ ;\ x_1x_2 \cdots\ ).$$

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] = \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] = \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

$$(\cdots x_{i-1} \; s_i s_{1+1} \cdots s_k \; x_{k+1} \cdots) \in C_i[s_i \ldots s_k]$$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define a probability measure by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ldots \mu(C_k[s_k]) = p_{s_i} \ldots p_{s_k}$.

$(\Sigma, \mathcal{C}, \mu)$ is a probability space.

A measurable map $T : X \to X$ is a *Bernoulli transformation* if it is isomorphic to a Bernoulli shift.

Isomorphism problem of Bernoullis shifts

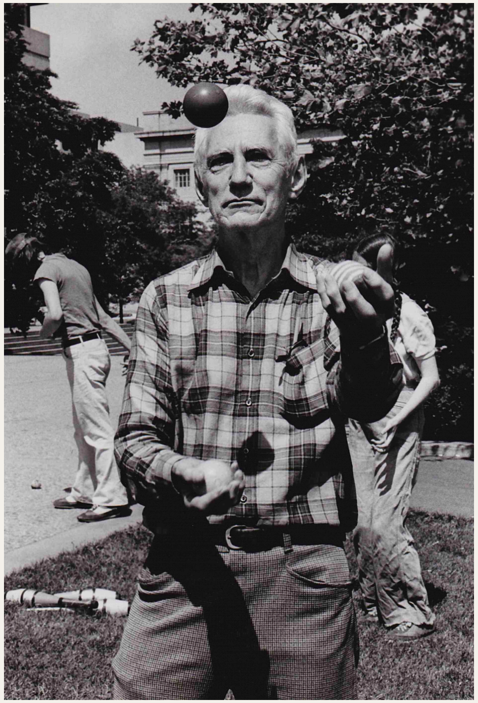
- Von Neumann: spectral isomorphism.
- Kolmogorov and Sinai entropy.
- Ornstein isomorphism theorem.

Shannon Entropy

For a probability distribution $\rho = (p_\alpha : \alpha \in \mathrm{A})$,

$$h(\rho) \stackrel{\text{def}}{=} \sum_{\alpha \in \mathrm{A}} -p_\alpha \log p_\alpha.$$

Entropy is the measure of uncertainty.



Claude Shannon

Kolmogorov-Sinai Entropy

Entropy of a partition,

$$H_\mu(\mathcal{P}) \overset{\text{def}}{=} \sum_{P \in \mathcal{P}} -\mu(P) \log \mu(P).$$

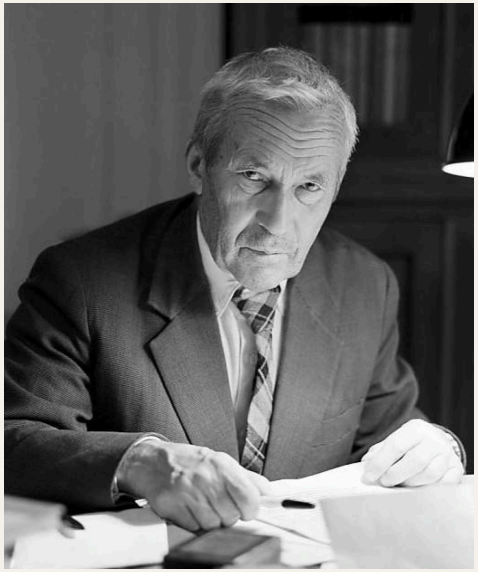
Entropy of a transformation with respect to a partition,

$$h_\mu(T, \mathcal{P}) \overset{\text{def}}{=} \lim_{k \to \infty} \frac{1}{k} H_\mu\left(\bigvee_{i=0}^{k-1} T^{-i} \mathcal{P} \right).$$

Entropy of a transformation,

$$h_\mu(T) \overset{\text{def}}{=} \sup_{\mathcal{P}} h_\mu(T, \mathcal{P}).$$

$T_1 \simeq T_2 \Rightarrow h_\mu(T_1) = h_\mu(T_2)$.



Andrei Kolmogorov

Kolmogorov-Sinai Entropy

Kolmogorov-Sinai theorem

Let $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$ to be a non-decreasing generating sequence of partitions with finite entropy. Then,

$$h_\mu(T) = \lim_k h_\mu(T, \mathcal{P}_k).$$

Yakov Sinai

Ornstein isomorphism theorem

> *Ornstein, 1970*
>
> *Bernoulli shifts with the same entropy are isomorphic.*

- Bôcher Memorial Prize
- Elect to American National Academy of Sciences
- Elect to American Academy of Arts and Sciences

Donald Ornstein, 1961

Non-invertible case: Extended Symbolic Dynamics

Encoding n-to-1 baker's transformations

Folding entropy for extended shifts

Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations

Encoding n-to-1 baker's maps

Mehdipour, P., Martins, N.

Archiv der Mathematik.

119, 199–211, (2022).

Zip shifts

- A and B be two alphabets with $|\mathrm{A}| \geq |\mathrm{B}|$
- $\varkappa : \mathrm{A} \to \mathrm{B}$ a surjective map
- Σ the space of all sequence of letters

$$(x_n)_{n\in\mathbb{Z}} = (\cdots x_{-2}x_{-1} \, ; \, x_0 x_1 \cdots)$$

$$\text{with } x_{-1}, x_{-2}, \ldots \in \mathrm{B} \text{ and } x_0, x_1, \ldots \in \mathrm{A}.$$

The (full) *zip shift* map is $\sigma_\varkappa : \Sigma \to \Sigma$ with

$$\sigma_\varkappa(\cdots x_{-1} \, ; \, x_0 x_1 \cdots) = (\cdots x_{-1}\varkappa(x_0) \, ; \, x_1 x_2 \cdots).$$

Zip shift space

Let $\mathcal{C}$ the σ-algebra generated by the cylinder sets

- $C_i[s] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s\}$
- $C_i[s_i \ldots s_k] \stackrel{\text{def}}{=} \{(x_n) \in \Sigma : x_i = s_i, \ldots, x_k = s_k\}$

Given a probability distribution $(p_\alpha : \alpha \in \mathrm{A})$ in A, we define $(p_\beta : \beta \in \mathrm{B})$

$$p_\beta \stackrel{\text{def}}{=} \sum_{\alpha \in \kappa^{-1}(\beta)} p_\alpha.$$

The measure μ is defined by

- $\mu(C_i[s]) = p_s$
- $\mu(C_i[s_i \ldots s_k]) = \mu(C_i[s_i]) \ldots \mu(C_k[s_k]) = p_{s_i} \ldots p_{s_k}.$

$(\Sigma, \mathcal{C}, \mu)$ is the *zip shift space.*

A map is a *LM-Bernoulli transformation* if is isomorphic to a zip shift map. A LM-Bernoulli with $m = |\mathrm{A}|, l = |\mathrm{B}|$ is called a (m, l)-Bernoulli transformation.

Zip shifts

PROP.3.6-11

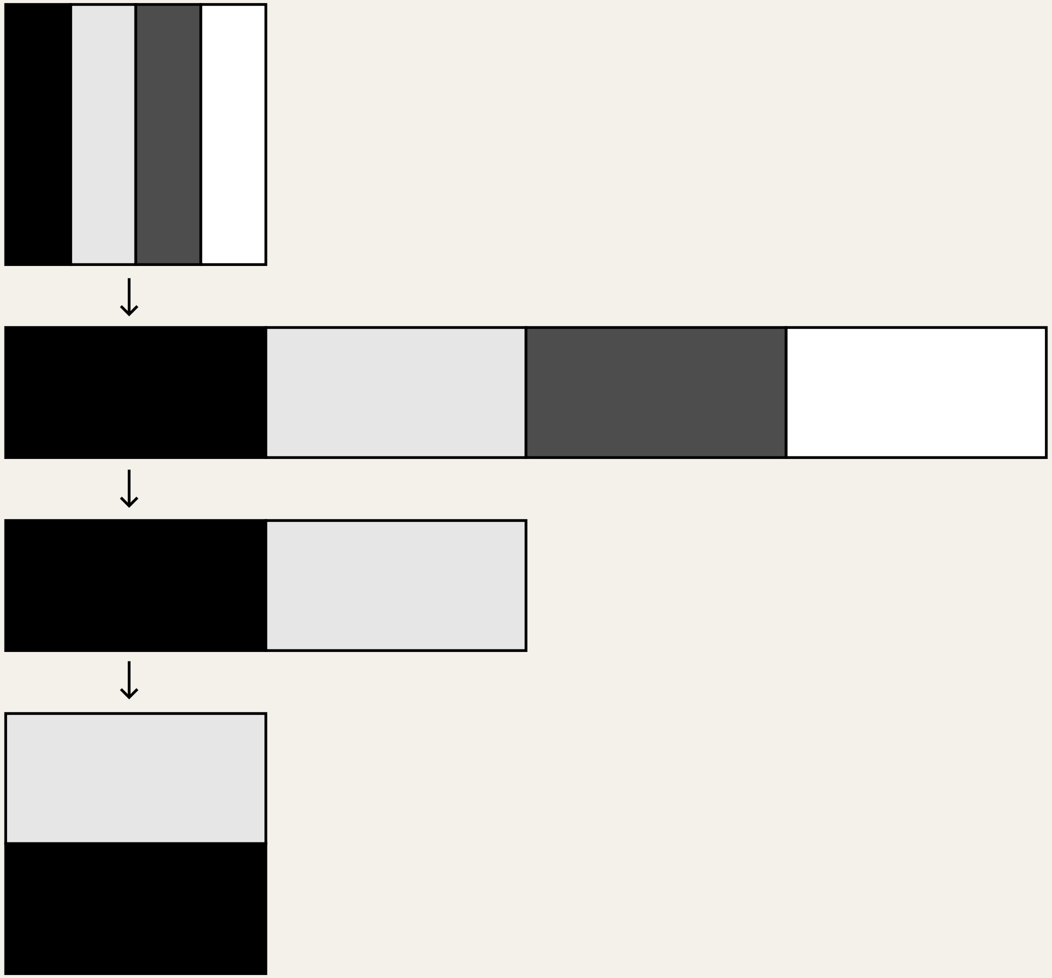
- σ_κ is a local homeomorphism
- σ_κ preserves the measure μ
- σ_κ is mixing and ergodic
- σ_κ has density of periodic points

The n-to-1 baker's maps

$T : [0,1]^2 \to [0,1]^2$ given by

$$T(x,y) = \begin{cases} \left(2nx, \frac{1}{2}y\right) & \text{if } \ 0 \leq x < \frac{1}{2n} \\ \left(2nx - 1, \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \ \frac{1}{2n} \leq x < \frac{2}{2n} \\ \left(2nx - 2, \frac{1}{2}y\right) & \text{if } \ \frac{2}{2n} \leq x < \frac{3}{2n} \\ \qquad \vdots & \ \vdots \qquad \vdots \\ \left(2nx - (2n-1), \frac{1}{2}y + \frac{1}{2}\right) & \text{if } \ \frac{2n-1}{2^n} \leq x \leq 1. \end{cases}$$

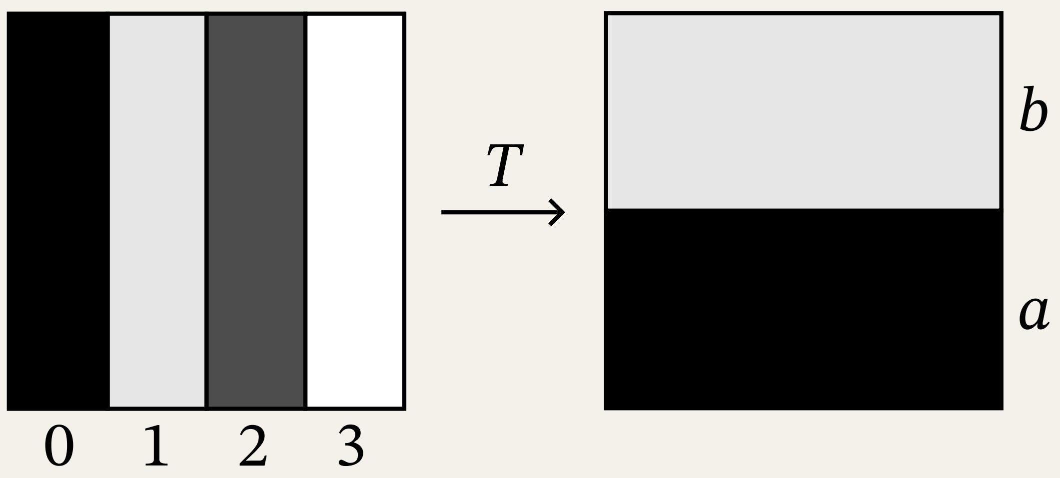
The n-to-1 baker's maps



The n-to-1 baker's maps are LM-Bernoulli

Theorem A THM.3.13

The n-to-1 baker's map is a $(2, 2n)$-Bernoulli transformation.



The n-to-1 baker's maps are chaotic

Theorem B Thm.3.13

The n-to-1 baker's map $\overline{T} : \overline{X} \to \overline{X}$ is chaotic in the sense of Devaney.

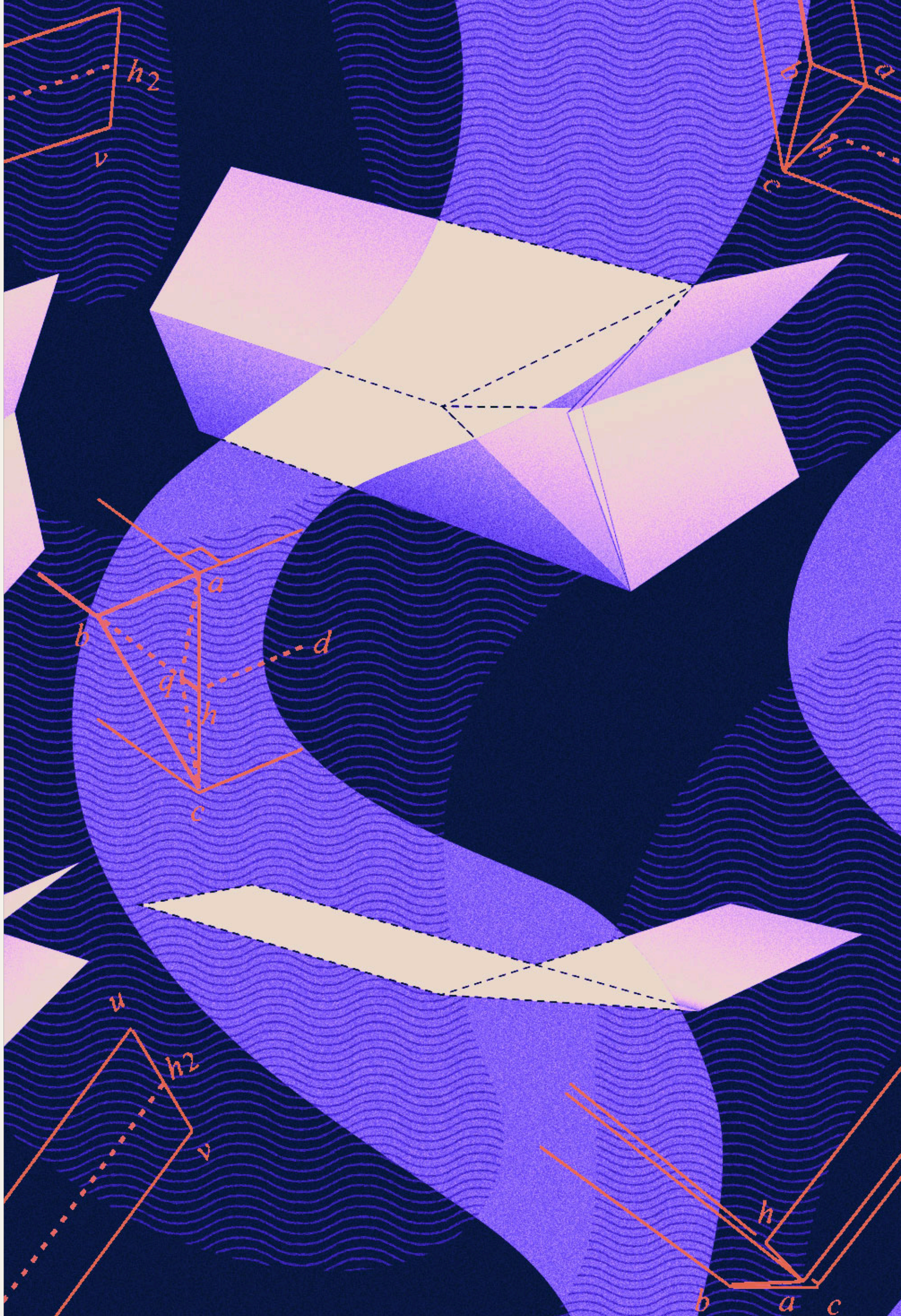
Devaney's chaos:

- Topologically transitive
- Density of periodic points
- Sensitive dependence on initial conditions.

Folding entropy of Extended Shifts

Martins, N., Mattos, P.G., Varão, R.

arXiv:2407.01828 (2024). 🔖



Kristina Armitage/Quanta Magazine

Partitions by cylinders

$$\mathcal{C}_i = \left\{ {}_i \right\} = \begin{cases} \{C_i[\alpha] : \alpha \in \mathrm{A}\} & \text{if } i \geq 0 \\ \{C_i[\beta] : \beta \in \mathrm{B}\} & \text{if } i < 0 \end{cases}$$

$$\mathcal{C}_{k_0 \cdots k_1} = \left\{ {}_{k_0} \cdots {}_{k_1} \right\} = \left\{ C_{k_0 \cdots k_1}\left[s_{k_0} \cdots s_{k_1} \right] : s_{k_0}, \ldots, s_{k_1} \in \mathrm{A} \cup \mathrm{B} \right\} = \bigvee_{i=k_0}^{k_1} \mathcal{C}_i.$$

Partitions by cylinders

- $\mathcal{C}_i$ and $\mathcal{C}_j$ are independent partitions, $\forall i, j$
- $\sigma_\kappa^i(\mathcal{C}_0) = \mathcal{C}_{-i}, \quad \forall i \geq 0$
- $\bigvee_{i=0}^{k-1} \sigma_\kappa^{-i}(\mathcal{C}_0) = \mathcal{C}_{0\cdots k-1}$
- $\bigvee_{i=-k}^{k-1} \sigma_\kappa^{-i}(\mathcal{C}_0) = \mathcal{C}_{-k\cdots k-1}$

Kolmogorov-Sinai entropy of zip shifts

Lem.4.1

$$H_\mu(\mathcal{C}_0) = \sum_{\alpha \in \mathrm{A}} -p_\alpha \log p_\alpha = h(\rho_{\mathrm{A}})$$

$$H_\mu(\mathcal{C}_{-1}) = \sum_{\beta \in \mathrm{B}} -p_\beta \log p_\beta = h(\rho_{\mathrm{B}})$$

Kolmogorov-Sinai entropy of zip shifts

LEM.4.2

$$H_\mu\left(\mathcal{C}_{-k_0\cdots k_1}\right) = k_0 H_\mu(\mathcal{C}_{-1}) + k_1 H_\mu(\mathcal{C}_0), \quad \forall k_0, k_1 \in \mathbb{N}$$

Kolmogorov-Sinai entropy of zip shifts

Lem.4.3

$$\mathcal{P}_k := \mathcal{C}_{-k\ldots k-1} \Rightarrow \bigvee_{i=0}^{n-1} \sigma_\kappa^{-i}\mathcal{P}_k = \mathcal{C}_{-k\cdots k+n-2}.$$

Kolmogorov-Sinai entropy of zip shifts

$$\begin{aligned} h_\mu(\sigma_\kappa, \mathcal{P}_k) &= \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} \sigma_\kappa^{-i} \mathcal{P}_k\right) \\ &= \lim_{n\to\infty} \frac{1}{n}\big(n\, H_\mu(\mathcal{C}_{-1}) + (k+n-1) H_\mu(\mathcal{C}_0)\big) \\ &= H_\mu(\mathcal{C}_0) \end{aligned}$$

Kolmogorov-Sinai entropy of zip shifts

Theorem C THM.4.4

$$h_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0).$$

In fact, $\{\mathcal{P}_k\}$ is a generating sequence and is non-decreasing. Then, by the Kolmogorov-Sinai theorem,

$$h_\mu(\sigma_\kappa) = \lim_{k\to\infty} h_\mu(\sigma_\kappa, \mathcal{P}_k) = H_\mu(\mathcal{C}_0).$$

Folding entropy

$$\mathcal{F}_\mu(T) \coloneqq H_\mu\bigl(\mathcal{E}/T^{-1}\mathcal{E}\bigr).$$

$$H_\mu(\mathcal{P}/\mathcal{R}) \coloneqq \int_{R\in\mathcal{R}} H_{\mu_R}(\mathcal{P}/R)\,\mu_R(dR),$$

where $\{\mu_R\}_{R\in\mathcal{R}}$ is a disintegration of μ with respect to $\mathcal{R}$.

David Ruelle

Folding entropy of zip shifts

Theorem D Thm.4.6

$$\mathcal{F}_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).$$

Disintegration of the measure

- $\hat{x}(\alpha) \coloneqq (\cdots x_{-2}\,;\, \alpha x_0 \cdots), \quad \forall \alpha \in \kappa^{-1}(x_{-1})$
- $\hat{x} \coloneqq \sigma_\kappa^{-1}(x) = \left\{\hat{x}(\alpha) : \alpha \in \kappa^{-1}(x_{-1})\right\}$
- $\hat{X} \coloneqq \{\hat{x} : x \in X\} \subset \sigma_\kappa^{-1}(\mathcal{E}),\ X \subset \Sigma.$
- The *quotient measure* $\hat{\mu}$ is given by

  $$\hat{\mu}(\hat{X}) = \mu\big(\pi^{-1}(\hat{X})\big) = \mu\big(\sigma_\kappa^{-1}(X)\big) = \mu(X).$$

Disintegration of the measure

- For every $\beta \in \mathrm{B}$, we define the following probability distribution

$$\left(q^{\beta}_{\alpha} : \alpha \in \kappa^{-1}(\beta)\right), \text{where } q^{\beta}_{\alpha} := \frac{p_{\alpha}}{p_{\beta}}.$$

- The *conditional measure* on $\hat{x} \in \sigma_{\kappa}^{-1}(\varepsilon)$ is given by

$$\mu_{\hat{x}}(\{\hat{x}(\alpha)\}) := q_{s}^{x_{-1}}, \ \forall \alpha \in \kappa^{-1}(x_{-1}).$$

The family $\{\mu_{\hat{x}}\}_{\hat{x} \in \sigma_{\kappa}^{-1}(\varepsilon)}$ is a *disintegration* of μ with respect to $\sigma_{\kappa}^{-1}(\mathcal{E})$.

Folding entropy of zip shifts

The folding entropy of σ_κ is given by

$$\mathcal{F}_\mu(\sigma_\kappa) \stackrel{\text{def}}{=} H_\mu\big(\mathcal{E}/\sigma_\kappa^{-1}(\mathcal{E})\big) = \int_{\hat{x}\in\sigma_\kappa^{-1}(\mathcal{E})} H_{\mu_{\hat{x}}}(\mathcal{E}/\hat{x})d\hat{\mu}(\hat{x})$$

$$
\begin{aligned}
\mathcal{F}_\mu(\sigma_\kappa) &= \sum_{\beta\in\mathrm{B}} \int_{\hat{x}\in\hat{C}_{-1}[\beta]} H_{\mu_{\hat{x}}}(\mathcal{E}/\hat{x})d\hat{\mu}(\hat{x}) \\
&= \sum_{\beta\in\mathrm{B}} \sum_{\alpha\in\kappa^{-1}(\beta)} \left(-q_\alpha^\beta \log q_\alpha^\beta\right) \hat{\mu}(\hat{C}_{-1}[\beta]) \\
&= \sum_{\beta\in\mathrm{B}} \sum_{\alpha\in\kappa^{-1}(\beta)} \left(-q_\alpha^\beta \log q_\alpha^\beta\right) p_\beta \\
&= \sum_{\beta\in\mathrm{B}} \sum_{\alpha\in\kappa^{-1}(\beta)} -p_\alpha(\log p_\alpha - \log p_\beta), \text{since } q_\alpha^\beta \cdot p_\beta = p_\alpha \\
&= \sum_{\alpha\in\mathrm{A}} -p_\alpha \log p_\alpha - \sum_{\beta\in\mathrm{B}} -p_\beta \log p_\beta = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1}).
\end{aligned}
$$

Folding entropy of zip shifts

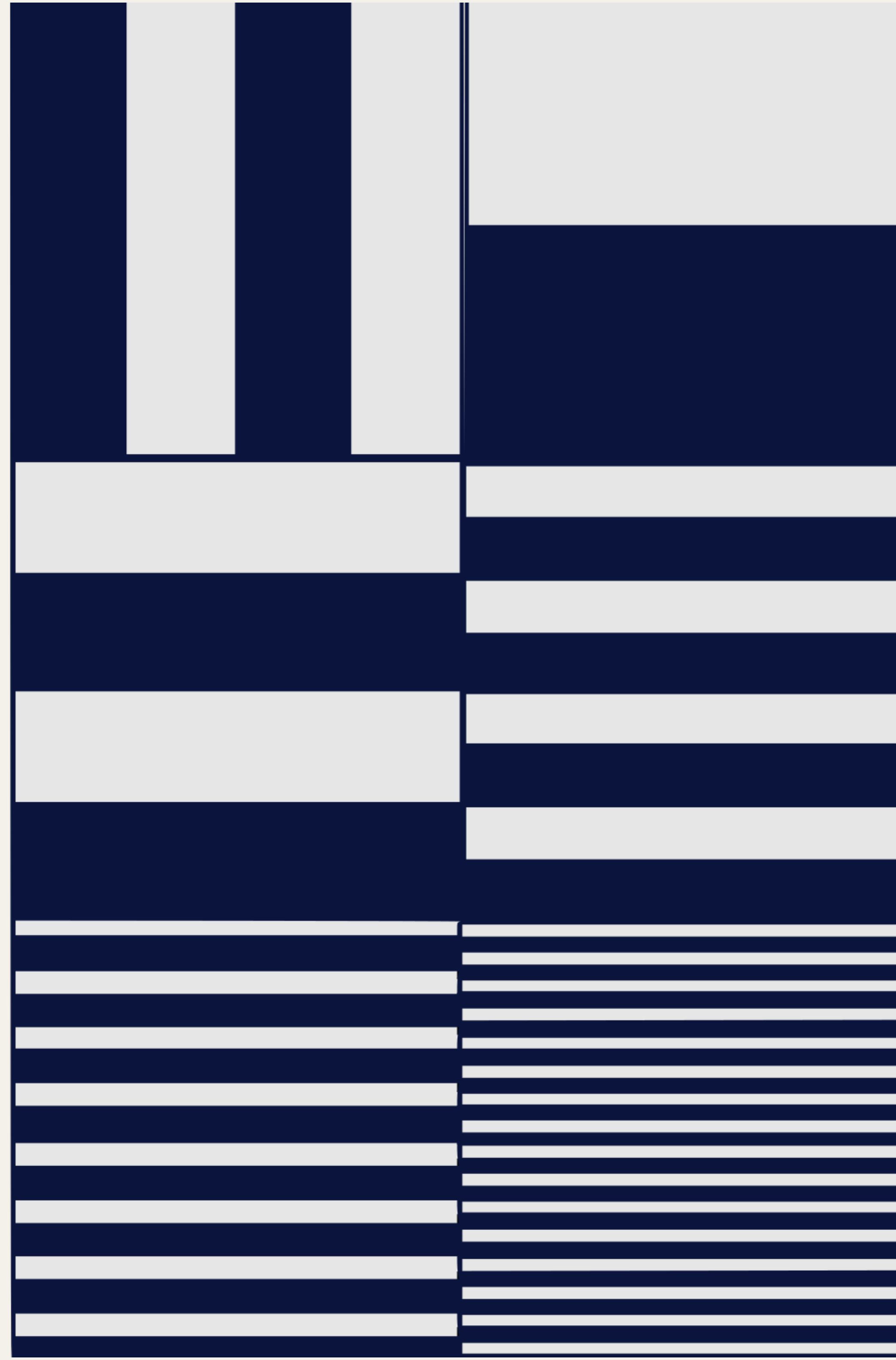
In particular,

$$h_\mu(\sigma_\kappa) = \mathcal{F}_\mu(\sigma_\kappa) + h(\rho_{\mathrm{B}}).$$

Ornstein isomorphism theorem for n-to-1 LM-Bernoulli transformations

Martins, N.,Mehdipour, P, Varão, R.

Preprint (2025) .



Isomorphism theorem

Theorem E THM.5.39

Two n-to-1 LM-Bernoulli transformations of same entropy are isomorphic.

Uniform zip shifts

LEM.5.2; PROP. 5.3; THM.5.4

- σ_κ n-to-1 $\Rightarrow$ σ_κ is a (k, kn)-zip shift
- $\sigma_{\kappa_1}, \sigma_{\kappa_2}$ uniform n-to-1 (k, kn)-zip shifts $\Rightarrow \sigma_{\kappa_1} \simeq \sigma_{\kappa_2}$
- Let $\sigma_{\kappa_1}, \sigma_{\kappa_2}$ uniform n-to-1 zip shifts. Then

$$\sigma_{\kappa_1} \simeq \sigma_{\kappa_2} \Leftrightarrow h_\mu(\sigma_{\kappa_1}) = h_\mu(\sigma_{\kappa_2})$$

Ornstein characterization of Bernoulli shifts

Ornstein, 1974 PROP.5.5

An automorphism $T : X \to X$ is isomorphic to a Bernoulli shift $\sigma : \Sigma_\mathrm{A} \to \Sigma_\mathrm{A}$ with distribution $\rho_\mathrm{A} = (p_\alpha : \alpha \in \mathrm{A})$ if, and only if, there is a partition $\mathcal{P}$ such that

a) $\mathrm{dist}(\mathcal{P}) = \rho_\mathrm{A}$

b) $\mathcal{P}$ is a generating for T

c) $\left\{T^k \mathcal{P}\right\}_{k \in \mathbb{N}}$ is a independent sequence.

Ornstein characterization of Bernoulli shifts

Ornstein, 1974 PROP.5.6

Two Bernoulli transformations are isomorphic if, and only if, there are partitions $\mathcal{P}$ and $\mathcal{R}$ such that

$$\operatorname{dist}\left(\bigvee_{i=0}^{k} T_1^{-i}\mathcal{P}\right) = \operatorname{dist}\left(\bigvee_{i=0}^{k} T_2^{-i}\mathcal{R}\right), \quad \forall k \in \mathbb{N}.$$

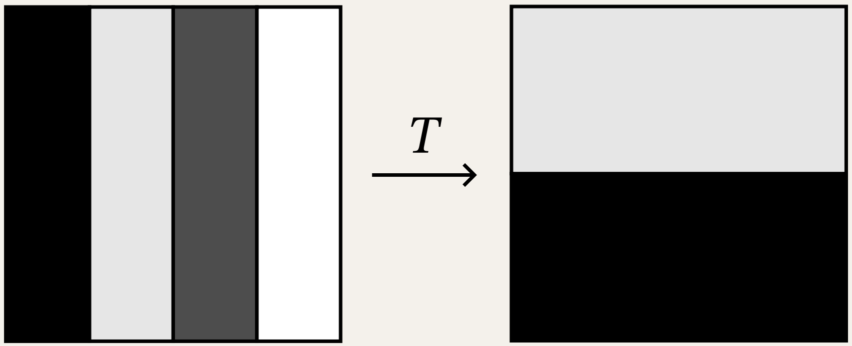
Domain and image partitions

- A image partition $\mathcal{Q} = \{Q_1, \ldots, Q_m\}$ of a n-to-1 local isomorphism a partition such that for all $P_i \in T^{-1}Q_j$, the map

$$T_{|_{P_i}} : P_i \to X$$

  is an automorphism.

- The collection $\mathcal{P}$ of all P_i is a domain partition



Characterization of n-to-1 LM Bernoulli

THM.5.19

An n-to-1 local isomorphism $T : X \to X$ is a LM-Bernoulli transformation with distribution $\rho_{\mathrm{A}} = (p_\alpha : \alpha \in \mathrm{A})$ if, and only if, there is a domain partition $\mathcal{P}$ such that

a) $\mathrm{dist}(\mathcal{P}) = \rho_{\mathrm{A}}$

b) $\mathcal{P}$ is a generating for T

c) The sequences $\left\{T^k \mathcal{P}\right\}_{k\in\mathbb{N}}$ and $\left\{T^{-k} \mathcal{P}\right\}_{k\in\mathbb{N}}$ are independent.

The copying condition

Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and $\mathcal{P}$ and $\mathcal{R}$ be partitions of X_1 and X_2, respectively.

The *process* $(T_1, \mathcal{P})$ is a *copy* of the process $(T_2, \mathcal{R})$, and we denote by

$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R})$$

when, for all $k \geq 0$,

$$\operatorname{dist}\left(\bigvee_{-k}^{k} T_1^{-i}\mathcal{P}\right) = \operatorname{dist}\left(\bigvee_{-k}^{k} T_2^{-i}\mathcal{R}\right).$$

The copying condition

THM.5.21

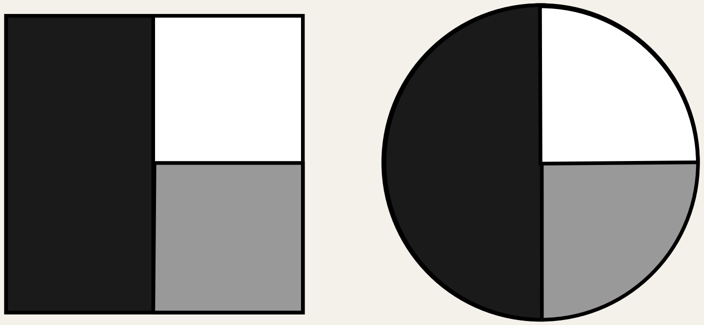
Let T_1, T_2 to be two n-to-1 LM-Bernoulli transformations and $\mathcal{P}$ and $\mathcal{R}$ to be generating partitions, respectively. Then,

$$(T_1, \mathcal{P}) \sim (T_2, \mathcal{R}) \Leftrightarrow T_1 \simeq T_2.$$

Metrics on partitions and processes

- Distance between the distributions of two partitions of same cardinality:

$$|\mathrm{dist}(\mathcal{P}) - \mathrm{dist}(\mathcal{R})| = \sum_{i=1}^{k} |\mu(P_i) - \mu(R_i)|.$$

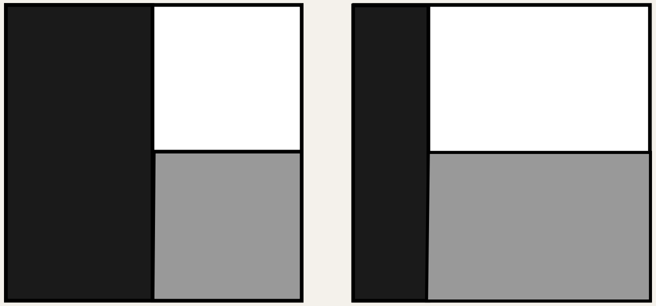


$$\mathrm{dist}(\mathcal{P}) = \mathrm{dist}(\mathcal{R}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

Metrics on partitions and processes

- Distance between partitions of the same space and same cardinality:

$$|\mathcal{P} - \mathcal{R}| = \sum_{i=1}^{k} \mu(P_i \triangle R_i)$$



$$\mathrm{dist}(\mathcal{P}) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

$$\mathrm{dist}(\mathcal{R}) = \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$$

$$|\mathcal{P} - \mathcal{R}| = \frac{1}{2}.$$

- $\mathcal{P} \mapsto h_\mu(T, \mathcal{P})$ is a continuous function in the partition metric.
- The space of all partitions is connected in the partition metric.

Metrics on partitions and processes

- Distance between sequences of partitions

$$d\left(\{\mathcal{P}_i\}_1^k, \{\mathcal{R}_i\}_1^k\right) = \inf \frac{1}{k} \sum_{i=1}^{k} \left|\overline{\mathcal{P}}_i - \overline{\mathcal{R}}_i\right|,$$

where the infimum is taken over all sequences of partitions $\left\{\overline{\mathcal{P}}_i\right\}_1^k, \left\{\overline{\mathcal{R}}_i\right\}_1^k$ of a same Lebesgue space such that

$$\operatorname{dist}\left(\bigvee_{i=0}^{k} \overline{\mathcal{P}}_i\right) = \operatorname{dist}\left(\bigvee_{i=0}^{k} \mathcal{P}_i\right), \quad \operatorname{dist}\left(\bigvee_{i=0}^{k} \overline{\mathcal{R}}_i\right) = \operatorname{dist}\left(\bigvee_{i=0}^{k} \mathcal{R}_i\right)$$

Metrics on partitions and processes

- Distance between processes

$$d((T_1, \mathcal{P}), (T_2, \mathcal{R})) = \sup_k \mathrm{d}\left(\left\{T_1^{-i}\mathcal{P}\right\}_1^k, \left\{T_2^{-i}\mathcal{R}\right\}_1^k\right)$$

Finitely determined processes

An n-to-1 LM-Bernoulli process $(T_1, \mathcal{P})$ is *finitely determined* if for every $\varepsilon > 0$, there are $\delta > 0$ and $k \in \mathbb{N}$ such that if an n-to-1 LM-Bernoulli process $(T_2, \mathcal{R})$ satisfies the conditions

a) $|\mathcal{P}| = |\mathcal{R}|$

b) $\left|h_\mu(T_1, \mathcal{P}) - h_\mu(T_2, \mathcal{R})\right| < \delta$

c) $\left|\mathrm{dist}\left(\bigvee_{i=0}^{k} T_1^{-1}\mathcal{P}\right) - \mathrm{dist}\left(\bigvee_{i=0}^{k} T_2^{-1}\mathcal{R}\right)\right| < \delta$,

then

$$d((T_1, \mathcal{P}), (T_2, \mathcal{R})) < \varepsilon.$$

Cor.5.24

If $(T,\mathcal{P})$ is an n-to-1 LM-Bernoulli process and $\left\{T^{-i}\mathcal{P}\right\}_{i\in\mathbb{N}}$ is an independent sequence, then $(T,\mathcal{P})$ is finitely determined.

Rokhlin lemma for LM-Bernoulli THM.5.25

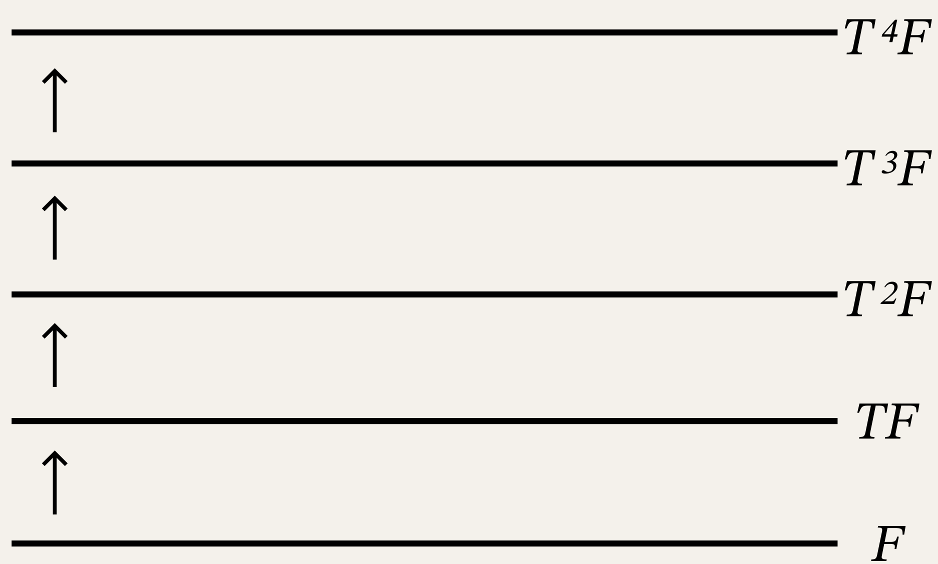
Let $T : X \to X$ be a LM-Bernoulli transformation, $k \geq 0$ and $\varepsilon > 0$. There is a disjoint measurable sequence

$$F, TF, \ldots, T^{k-1}F,$$

such that $\mu\left(\bigcup_{i=0}^{k-1} T^i F\right) > 1 - \varepsilon$.

Stacks and Gadgets

The sequence $\left\{T^iF\right\}_0^{k-1}$ is a *stack* of base F and lenght k.



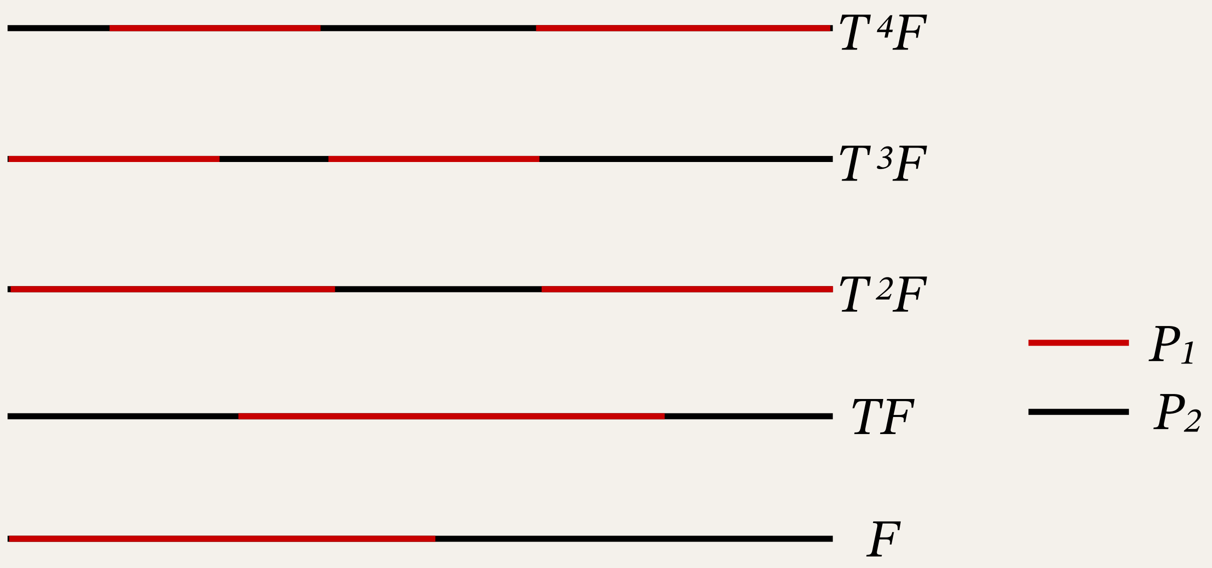
Strong Rokhlin lemma for LM-Bernoulli THM.5.27

Let $(T, \mathcal{P})$ to be a LM-Bernoulli process, $k \geq 0$ and $\varepsilon > 0$. There is a stack

$$F, TF, \ldots, T^{k-1}F,$$

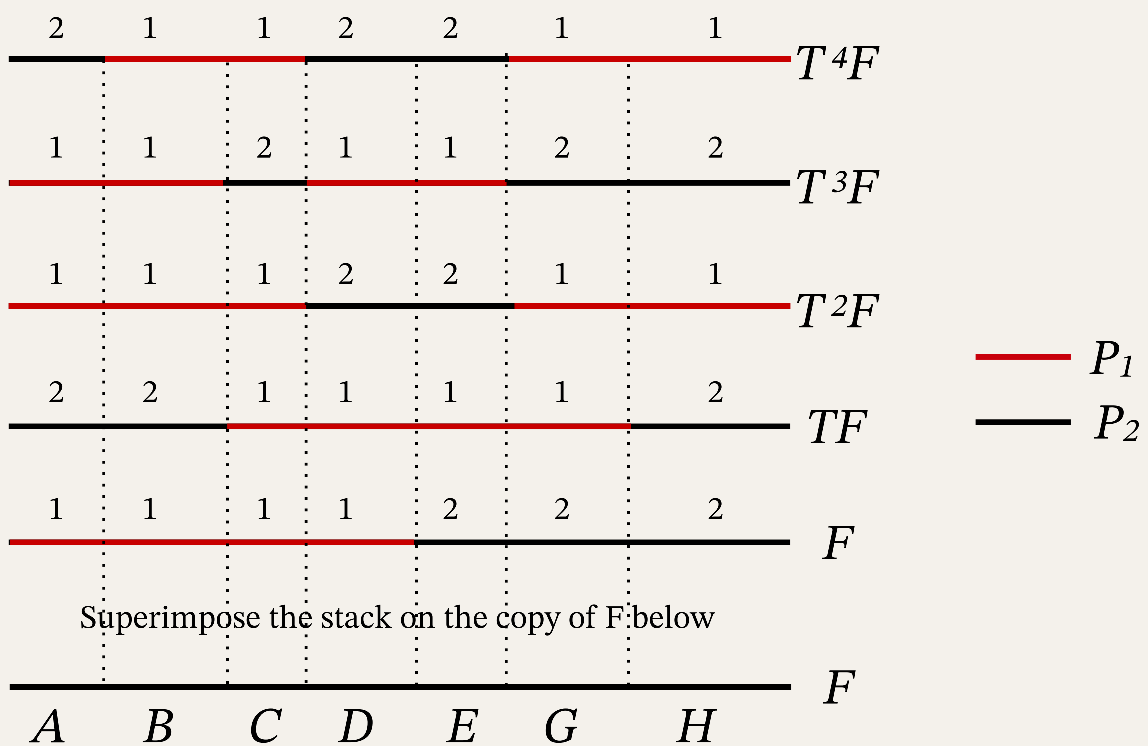
such that $\mu\left(\bigcup_{i=0}^{k-1} T^i F\right) > 1 - \varepsilon$ and $\operatorname{dist}(\mathcal{P}/F) = \operatorname{dist}(\mathcal{P})$.

Stacks and Gadgets



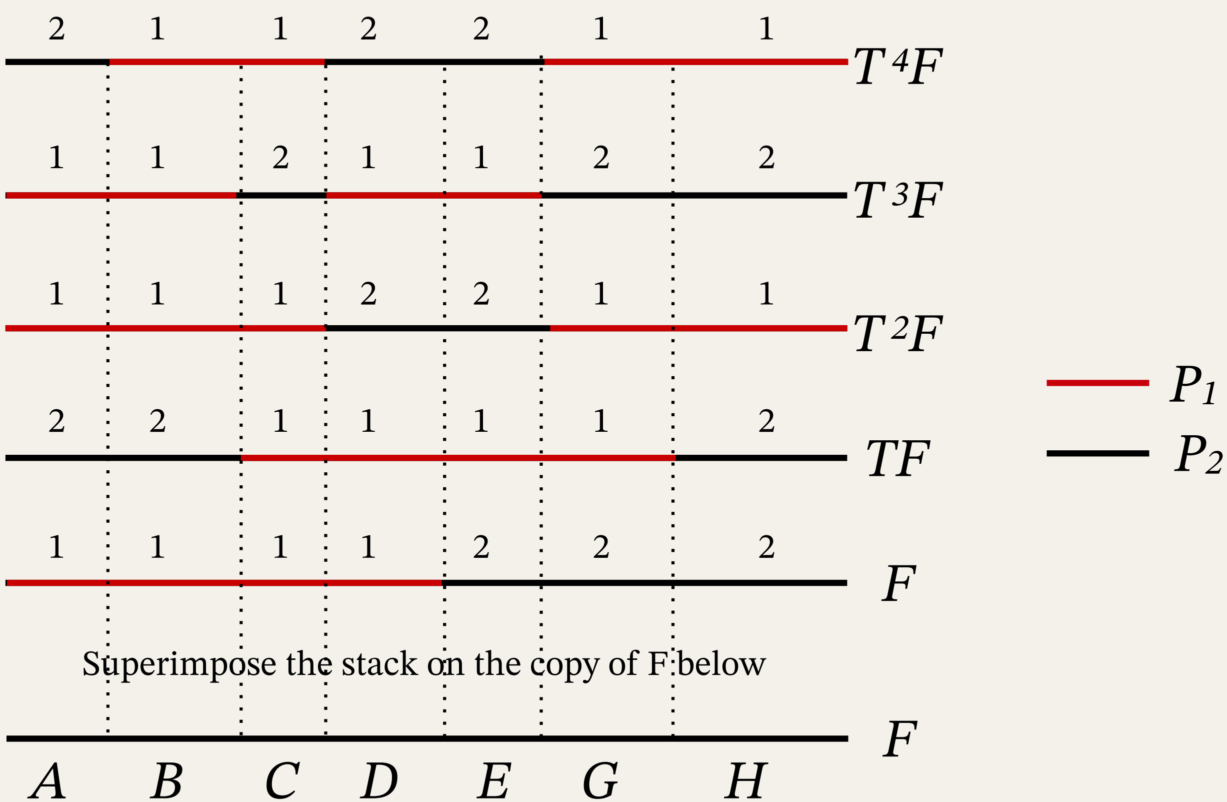
The induced distribution of the base of the stack is the same as $\mathcal{P}$.

Stacks and Gadgets



$$\bigvee_{i=0}^{5-1} T^{-i}\big(\mathcal{P}/T^{i}F\big)/F = \bigvee_{i=0}^{5-1} T^{-i}\mathcal{P}/F.$$

Stacks and Gadgets



$$A = F \cap P_1 \cap T^{-1}(P_2) \cap T^{-2}(P_1) \cap T^{-3}(P_1) \cap T^{-4}(P_2).$$

$\mathcal{P}$-5-name of $A : (1, 2, 1, 1, 2)$

Getting a copy

PROP. 5.35

Let $(T_1, \mathcal{P})$ and $(T_2, \mathcal{R})$ to be two n-to-1 LM-Bernoulli processes with $(T_2, \mathcal{R})$ f.d. Given $\varepsilon > 0$, there are $\delta > 0$ and $k \in \mathbb{N}$ such that if:

a) $h_\mu(T_1) \geq h_\mu(T_2, \mathcal{R})$

b) $|\mathcal{P}| = |\mathcal{R}|$

c) $\left|\mathrm{dist}\left(\bigvee_{i=0}^{k-1} T_1^{-i}\mathcal{P}\right) - \mathrm{dist}\left(\bigvee_{i=0}^{k-1} T_2^{-i}\mathcal{R}\right)\right| < \delta$

d) $\left|h_\mu(T_1, \mathcal{P}) - h_\mu(T_2, \mathcal{R})\right| < \delta$

then there is $\overline{\mathcal{P}}$ such that $\left|\overline{\mathcal{P}} - \mathcal{P}\right| < \varepsilon$ and $\left(T_1, \overline{\mathcal{P}}\right) \sim (T_2, \mathcal{R})$.

Lem. 5.36

$\mathcal{P}$ is a generating partition for T iff for each $\mathcal{R}$ and $\varepsilon > 0$, there is k such that

$$\mathcal{R} \underset{\varepsilon}{\prec} \bigvee_{-k}^{k} T^i \mathcal{P}.$$

Getting a better copy

LEM. 5.37

Let $\mathcal{P}$ be a generating partition for a n-to-1 LM-Bernoulli T, and suppose

$$h_\mu(T,\mathcal{P}) = h_\mu(T,\mathcal{R})$$

with $(T,\mathcal{P}),(T,\mathcal{R})$ both f.d. Given $\varepsilon > 0$, there is $\overline{\mathcal{R}}$ such that

a) $\left(T,\overline{\mathcal{R}}\right) \sim (T,\mathcal{R})$

b) $\left|\overline{\mathcal{R}} - \mathcal{R}\right| < \varepsilon$

c) $\mathcal{P} \underset{\varepsilon}{\prec} \bigvee_{-k}^{k} T^i\overline{\mathcal{R}},\ k \in \mathbb{N}$

Getting a better copy

PROP. 5.38

Let $\mathcal{P}$ be a generating partition for a n-to-1 LM-Bernoulli T, and suppose

$$h_\mu(T,\mathcal{P}) = h_\mu(T,\mathcal{R})$$

with $(T,\mathcal{P}),(T,\mathcal{R})$ both f.d. Given $\varepsilon > 0$, there is $\overline{\mathcal{R}}$ such that

a) $\left(T,\overline{\mathcal{R}}\right) \sim (T,\mathcal{R})$

b) $\left|\overline{\mathcal{R}} - \mathcal{R}\right| < \varepsilon$

c) $\overline{\mathcal{R}}$ is generating for T.

Getting a better copy

> PROP. 5.38
>
> Let $(T_1, \mathcal{P})$ and $(T_2, \mathcal{R})$ be two n-to-1 LM-Bernoulli processes f.d, $\mathcal{P}$ and $\mathcal{R}$ generating partitions, such that $h_\mu(T_1) = h_\mu(T_2)$. Then T_1 and T_2 are isomorphic.

- $(T_2, \mathcal{R})$ f.d $\to$ choose $\mathcal{P}'$ near to $\mathcal{P}$ such that $(T_1, \mathcal{P}') \sim (T_2, \mathcal{R})$.
- $h_\mu(T, \mathcal{P}) = h_\mu(T, \mathcal{R}), \mathcal{P}$ generating $\to$ choose a generating $\overline{\mathcal{P}}$ near to $\mathcal{P}'$ such that

$$\left(T_1, \overline{\mathcal{P}}\right) \sim (T_2, \mathcal{R}).$$

Conclusion

Overview

Thm A The n-to-1 baker's map are $(2, 2n)$-Bernoulli.

Thm B The n-to-1 baker's map $\overline{T}$ is chaotic.

Thm C $h_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0)$.

Thm D $\mathcal{F}_\mu(\sigma_\kappa) = H_\mu(\mathcal{C}_0) - H_\mu(\mathcal{C}_{-1})$.

Thm E Two n-to-1 LM-Bernoulli maps of same entropy are isomorphic.

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Thank you!